

Pseudolocalized Three-dimensional Solitary Waves as Quasi-Particles

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A higher-order dispersive equation is introduced as a candidate for the governing equation of a field theory. A new class of solutions of the three-dimensional field equation are considered, which are not localized functions in the sense of the integrability of the square of the profile over an infinite domain. For the new type of solutions, the gradient and/or the Hessian/Laplacian are square integrable. In the linear limiting case, analytical expression for the pseudolocalized solution is found and the method of variational approximation is applied to find the dynamics of the centers of the quasi-particles (QPs) corresponding to these solutions. A discrete Lagrangian can be derived due to the localization of the gradient and the Laplacian of the profile. The equations of motion of the QPs are derived from the discrete Lagrangian. The pseudomass (“wave mass”) of a QP is defined as well as the potential of interaction. The most important trait of the new QPs is that at large distances, the force of attraction is proportional to the inverse square of the distance between the QPs. This can be considered analogous to the gravitational force in classical mechanics.

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I. INTRODUCTION

The first half of the 20th Century oversaw the formation of the current paradigm in physics, in which particles and fields are the main distinct forms of matter. Then naturally arose the question of the interconnection between these two facets of the physical reality. In the early 1960s, Skyrme [1, 2] came up with the idea that a localized solution of a given field equation can be considered as a particle. Because of the formidable mathematical difficulties in multiple spatial dimensions, the new idea was demonstrated in 1D for the case of the *sine-Gordon* equation (sGE), for which an analytical solution was found in [2] for a profile composed by the superposition of two localized shapes. The shapes interacted (scattered) in a very similar fashion as two particles would do if they collided with each other. In a seemingly unrelated research, Fermi and coworkers [3] discovered numerically that a virtually random initial condition for the difference equations modeling an atomic lattice tends in the long term to organize into a chain of localized shapes. Observing that the Korteweg–de Vries equation (KdVE) is the limit of the difference equations modeling the lattice, Zabusky and Kruskal [4] performed similar numerical experiments for KdVE and confirmed the tendency of the initially “thermalized” modes to organize into localized waves that interact as particles: the term *soliton* was introduced for this kind of wave.

Nowadays, the subject of solitons enjoys considerable attention. The surveys [5, 6] give a good perspective of the wide scope of soliton research at the end of the 1970s. Applications to dynamics of atomic lattices are summarized in [7]. In [8] an excellent updated survey of the application of the soliton concept in elasticity can be

found. Several general surveys are also available [9–11]. The list of the equations for which soliton solutions are sought is ever expanding (see, e.g., [12]).

If an initial condition is composed by the superposition of two localized waves, the evolution of this wave system results in virtual recovery of the shapes of the two initial localized waves after their collision but in shifted relative positions after outgoing from the site of interaction (see [9, 11, 13, 14], among others). The collision property is what justifies calling these localized waves “quasi-particles” (QPs). A QP solution of a fully integrable system is a soliton in the strict sense. For the case of non-fully-integrable system, the term QP is safer, but many researchers use the term “soliton” in a broader sense that includes also the non-fully-integrable cases. When the physical system is described by equation(s) that conserve the energy and momentum, then the same conservation properties are inherited by the QPs, as would be necessary for actual subatomic particles.

What would be called a “two-soliton” solution in the late 1960s, was given in 1956 in [2] for the sGE, and the resemblance between the two-soliton solution and a specially constructed superposition of one-solitons could be seen in the cited paper. In [1], the localized solutions were qualitatively related to mesons assuming that the meson field is governed by the sGE. What is more important is that Skyrme introduced the fundamental notion that the localized (in an appropriate sense) solutions can be considered as particles of the field described by the particular equation under consideration. Using the techniques for finding analytically the two-soliton solution, an important analytical result was obtained in [15, 16] by extracting the positions of the centers of the individual solitons from the actual two-soliton profile. Thus the trajectories of Skyrme’s “particle” (the QP in the modern terminology) were found and shown to bend during the interaction. The last result completes, in a sense, Skyrme’s proposal that the dynamics of the localized solution of

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the respective field equation can describe subatomic particle interactions. The impressive analytical success here can be clearly attributed to the full integrability of sGE and to the 1D nature of the solution. This line of research is currently actively pursued with the goal of creating a kind of “multi-body” formalism abased on the connection to the field equations [17]. Thus, Skyrme’s idea to establish the relation between the dynamics of soliton solutions of a field equation and subatomic particles is a fruitful paradigm (see, e.g., [18]).

When the field equation is not fully integrable, the way to obtain information about the QP behavior of the solution is to make an ansatz that the two-soliton solution is fairly well approximated by the linear superposition of two one-soliton solutions with priory unknown trajectories, to derive a “discrete” Lagrangian with the trajectories as “generalized coordinates”, and to solve the Newtonian-like governing equations for the point dynamics of the centers of the individual solitons. This was proposed in [19, 20] and applied to the (non-integrable) nonlinear Klein-Gordon equation (KGE). The trajectory function for the center of a QP was called a “collective coordinate” in [20] (see, also, [21]) and collective variable (CV) [22]. Nowadays, this approach is known as *Variational Approximation* (VA). The success of the simplest approach with one CV inspired introducing more sophisticated VA models, some involving two (and more) CVs. An important step in this direction was made in [23], where the second CV is the “width” of the QP. Such a choice allows one to recover, as in [23, 24], the relativistic dynamics of a single QP in the case of KGE (see [25] for multiple CVs). Unfortunately, the generalization of the 2-CV method to 2-QP interaction is not trivial. For this reason the 2-CV approach was tested in [22] for the sGE case. The integrals for the potential of interaction are unsurmountable without some simplifying assumption (see [26]).

Despite this shortcoming, the VA is clearly a very promising way of establishing the wave-particle dualism for QPs, provided that some more pertinent models for the field equation are used. We would like to mention here that VA is one of the specific techniques to find an approximate solution of the field equations. The approach of reducing the system with distributed parameters to a discrete system amounts, in general, to a *Coarse Grain Description* (CGD) [27]. From this point of view the VA is the most physically consistent way to create a CGD of a continuous system, because it retains the main conservation laws related to the latter.

Before proceeding further, it is important to mention that recently, another avenue for further development of the VA has been opened. The full two-soliton solution shows a significant deformation of the wave profile when the two main solitons are close to each other. Such a deformation is clearly not contained in the mere superposition two one-soliton solutions. There is some utility to try to represent this deformation via the evolution of the widths of the QPS (as in the previous paragraph), but

there is also another very practical approach: to consider what is left from the two-soliton solution when the one-soliton solutions are subtracted as a new particle which is briefly born at the site of interaction and promptly dies after the separation of the main lumps (the so-called “ghost particles” [28]). This idea was elaborated in [29] and offers a very pertinent perspective on the world of elementary particle where the birth, death, and transmutation of the particles are incessant processes.

Apart from sGE and KGE, there are many different classes of dispersive nonlinear equations which possess localized solutions and for which the VA proved to be a very effective tool for approximate solutions. The nonlinear Schrödinger equation (NLS) has arisen in modeling the propagation of light along optical fibers [30]. Because of the nonlinearity, localized envelopes can appear on the carrier’s optical frequency, which behave like QPs. The phenomenology is much richer for NLS, and the solitons for the envelope can be ‘dark’ or ‘bright’ [6]. The bright solitons are sometime called ‘light bullets’ [31, 32] and play a very important role in discretizing (digitizing) the information carried through the fiber. The VA was applied to bright NLS solitons first in [33, 34] and since then NLS and vector NLS are the most important examples of the concept of QP. An excellent survey on this subject can be found in [6].

The concept of CGD of a field as an ensemble of QPs is clearly an important step in understanding the nature of wave-particle duality. Yet, there are essential differences between the currently known QPs and the real particles. The most conspicuous difference is that the QPs in 1D actually pass through each other because when the force is positive it remains positive up to the moment of collision, and then changes its sign. It is interesting to note that the 1D space is so restrictive that even the two-soliton solution has a form suggesting that the individual solitons passed through each other rather than scatter (see the in-depth discussion in [22]). It turns out [26] that only VA can shed light on what is happening, in the sense that the QPs scatter, but in this process because of the restricted freedom in 1D the scattered QP acquires the amplitude (pseudomass) of the other one, and the composite wave profile appears as if the QPs passed through each other. This effect was discovered in [35] and called “bouncing of mass” of the solitons for the case of two coupled KdVE. The repulsion or attraction of the QPs depends on the interaction potential defined by the asymptotic behavior of their tails (see [36], among others). The second difference between the QPs in 1D and the subatomic particles is that the attractive force decays exponentially with the distance between the QPs, while in reality, the attraction is given by Newton’s law of gravitation, for which the force decays with the inverse square of the distance between the centers of the particles. This makes it important to investigate QPs in dimensions higher than one, in hope of finding a potential of interaction that is more realistic. For this purpose any of the known generalized wave equations containing

dispersion should be a legitimate basic model.

The first 3D QPs were reported for the NLS in [31, 32], where the profiles of the light bullets are found numerically for the stationary QPs and then the VA is used to investigate their interaction. It is well known that the NLS is very fertile ground for experimenting with localized shapes, because of the envelope nature of the latter. It is not so straightforward a matter in the traditional field theories, such as *s*GE and KGE, because a solution of the type of the classical kink cannot be found in more than one dimension. Hence the simple notion of a lump of deformation of the field being a QP cannot be extended into 3D. Recently, oscillating localized 3D structures were found for the *s*GE in [37, 38] and called *compactons*. Forcing the solution of KGE to oscillate in time leads to the same equation for the 3D shape as in the above mentioned NLS works.

Despite their 3D nature, the localized solutions mentioned in the previous paragraph exhibit the same exponentially decaying tails (actually, slightly super-exponential), and the respective force acting between them will decay exponentially with the increase of the distance between the QPs.

The purpose of the present paper is to explore a new field equation and a new concept of localization of the solution with the aim of bringing some of the properties of QPs closer to the observed behavior of the objects in particle physics.

II. THE THIN SHELL AS A MODEL OF A FIELD

As already stated, in this paper we set to explore the possibility of using a novel nonlinear equation to describe a field. In doing this we would like to stay as close as possible to the spirit of quantum mechanics, i.e., as close as possible to the Schrödinger equation of wave mechanics, which has proven its relevance to the subject. As observed by Schrödinger himself [39] and elaborated in different papers of the present author (see [40] and the literature cited therein), if written for the real or imaginary part of the wave function, Schrödinger's equation belongs to the class of Euler–Bernoulli equations [41] describing the flexural deformations of a thin plate. While, the Euler–Bernoulli equation is the linear part of any shell/plate equation, the nonlinear terms depend on the assumptions about the nature of the deformations. In the theory of prolate deformations of shallow shells/plates, the von Kármán [42] equations are the current standard [43]. In the other extreme case of very steep deformations, the shell equation is akin to the Boussinesq equation, but the nonlinearity is different (see [40] and the literature cited therein). Since, the elementary particles have very small dimensions, the QPs that are supposed to describe them may exhibit even for small elevations deformations of order of unity and very large curvature (exactly the opposite to what is considered as a prolate

deformation). We choose the model from [40], which is essentially a non-prolate model, as our field equation, namely

$$u_{tt} = \Delta u - \alpha_3(\Delta u)^3 - \beta^2 \Delta^2 u, \quad (1)$$

where β^2 is the dispersion coefficient due to the stiffness of the shell (always positive) taken as the square of some parameter β for convenience. The coefficient α_3 gives the relative importance of the nonlinear term. The membrane tension (the coefficient of the Laplacian term) is taken to be equal to unity, because for the case of localized solutions over an infinite interval (no boundary conditions at specific points) it can be scaled out.

Now, the spherically-symmetric, stationary solution satisfies

$$\beta^2 \frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{dw}{dr} \right) - w + w^3 = 0, \quad (2a)$$

$$\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{du}{dr} \right) = w(r), \quad (2b)$$

where $-w$ is the curvature of the flexural deformation. Note that Eq. (2a) is the same as the one solved numerically in [32] and [38]. Because of the absence of biharmonic terms in these works, the result for w is the final result. For the time being, no analytical solution is available for this equation, and under the boundary condition $w(r) \rightarrow 0$ for $r \rightarrow \infty$ it has a solution which decays at infinity as e^{-r}/r . This solution is presented in the lower portion of Fig. 1. It corresponds to the *sech* solution in 1D and can be termed the “spherical *sech*”.

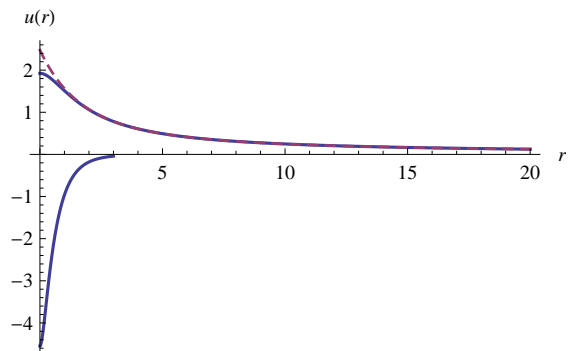


FIG. 1: The pseudo-localized solution (upper part) and the localized “spherical *sech*” (lower part) for $\beta = 1$. Solid lines: numerically computed profile; dashed line: best fit solution to the linearized problem given by $2.47546(1 - e^{-r})/r$.

After having the solution for w , the function u , which can be called “the wave function”, can be found by quadratures, namely

$$u(r) = \int_0^r \left[\frac{1}{r_1^2} \left(\int_0^{r_1} r_2^2 w(r_2) dr_2 \right) \right] dr_1 + \frac{C_1}{r} + C_2. \quad (3)$$

Here, one has to take $C_2 = 0$ in order that u decays at infinity. Respectively, $C_1 = 0$ ensures that there is

no singularity in the origin of the coordinate system. As suggested in [40], one can find a best-fit approximation to the numerical solution for w and then integrate it twice according to Eq. (3). The result is presented in the upper portion of Fig. 1. It is seen that the decay of the profile of u at infinity is as $1/r$, which is significantly slower than exponential decay.

The nonlinear terms are important for shaping the profile near the origin of the coordinate system (as can be seen in Fig. 1), but at larger distances the equation reduces to the following linear equation

$$u_{tt} = \Delta u - \beta^2 \Delta^2 u, \quad (4)$$

which is the celebrated Euler–Bernoulli equation. As shown in [40] this equation is akin to the Schrödinger equation, so interrogating Eq. (4) is a good place to start. The Lagrangian for Eq. (4) reads

$$L = \iiint_V \frac{1}{2} [(u_t)^2 - (\nabla u)^2 - \beta^2 (\Delta u)^2] d^3 \mathbf{x}, \quad (5)$$

where in the case under consideration the volume of integration V is the entire space. Without fear of confusion we will use in the integral the notation $-\infty$ and ∞ as the limits of the triple integral in what follows.

The linearized Eq. (4) has the solution (see [40])

$$\Phi(\rho) = \frac{1}{\rho} [1 - e^{-\rho/\beta}], \quad (6)$$

which is compared to the solution of the nonlinear equation in Fig. 1. One can see that the full nonlinear solution does not differ quantitatively from the linear solution given by Eq. (6), save the fact that the latter is not smooth at the origin as function of the variables x, y, z . It decays at infinity, but is not localized in the strict sense, because the integral of its square diverges linearly at infinity with the radius of the domain of integration. However, the integral of the square of its gradient converges:

$$\begin{aligned} \iiint_{-\infty}^{\infty} (\nabla \Phi)^2 dx dy dz &\equiv 4\pi \int_0^{\infty} \left[\frac{d\Phi(\rho)}{d\rho} \right]^2 \rho^2 d\rho \\ &= 4\pi \int_0^{\infty} \left[\frac{\exp(-\rho/\beta)}{\rho\beta} - \frac{1 - \exp(-\rho/\beta)}{\rho^2} \right]^2 \rho^2 d\rho \\ &= \frac{2\pi}{\beta}. \end{aligned} \quad (7)$$

As it should be expected, if the gradient is in $L^2(\mathbb{R}^3)$, the Laplacian belongs to the same space, in the sense that the integral of the square of the Laplacian converges too:

$$\begin{aligned} \iiint_{-\infty}^{\infty} [\Delta \Phi]^2 dx dy dz &\equiv 4\pi \int_0^{\infty} \left[\frac{1}{\rho^2} \frac{d}{d\rho} \rho^2 \frac{d\Phi(\rho)}{d\rho} \right]^2 \rho^2 d\rho \\ &= 4\pi \int_0^{\infty} \left[-\frac{\exp(-\rho/\beta)}{\rho\beta^2} \right]^2 \rho^2 d\rho = \frac{2\pi}{\beta^3}. \end{aligned} \quad (8)$$

Because of the quantitative closeness of the solution of the nonlinear equation to the linear one, the statements

about the convergence and/or non-convergence of the different integrals is the same for the solutions of the full and linearized equations. Following [40], we refer to this kind of solution as the *pseudolocalized* solution. The quantitative closeness to the linear solution allows us to investigate the interaction properties of the pseudolocalized QPs by analytical means using the result in Eq. (6).

It is interesting to observe here that in 2D, the solution of Eq. (4), with point symmetry, that does not exhibit a singularity at the origin is

$$u^{(2)}(r) = K_0(r/\beta) + \ln(r/\beta), \quad (9)$$

where K_0 is the modified Bessel function of the second kind. Clearly, this solution does not have a pole at the origin, but is not a localized function, because it increases logarithmically to infinity at $r \rightarrow \infty$. It is not even a pseudolocalized one in the sense of the definition of the present work, because the following integral diverges

$$\begin{aligned} \int_{\epsilon}^R \left(\frac{\partial u^{(2)}}{\partial r} \right)^2 r dr &= \int_{\epsilon}^R [(K'_0(\eta))^2 r + 2K'_0(\eta) + \frac{1}{r}] dr \\ &\propto \text{const} - \ln \epsilon + \ln R, \quad R \gg 1. \end{aligned} \quad (10)$$

This means that in 2D, pseudolocalized solutions of the Euler–Bernoulli equation as defined here do not exist.

III. LOCALIZED VERSUS PSEUDOLOCALIZED SOLUTIONS

Eq. (1) belongs to the generic class of Boussinesq equations (BEs). In different models of flows of ideal fluids with free surfaces or interfaces, different BEs arise. A generic version of Boussinesq’s equation can be written as

$$u_{tt} = \Delta[u - \alpha_2 u^2 - \alpha_3 u^3 - \beta^2 \Delta u]. \quad (11)$$

Eq. (11) is a generalized wave equation containing dispersion in the form of a biharmonic operator of the sought function. The dispersive effects of the biharmonic operator can be countered by the presence of a nonlinearity, and as a result a permanent wave of localized type may exist and propagate without change. This balance between the nonlinearity and dispersion was first found by Boussinesq [44] and we shall refer to it as the “Boussinesq paradigm”. In different physical situations leading to the same generic class of equations, the nonlinear terms can be different. In fluid mechanics, the most general form of the weakly-nonlinear approximation for flows in thin fluid layers is discussed in [45, 46].

Unfortunately, for the time being we are unable to find a two-soliton solution in the 3D case of equations of the type of Eq. (11), just as in 2D multi-soliton solutions are not known, except when the equation is simplified in one of the spatial directions (such as the so-called Kadomtsev–Petviashvili equation (KPE), see the original work [47] and general surveys [10, 13]).

The findings about the 2D localized solution of the Boussinesq equation can be summarized as follows. The profile of the soliton has a super exponential decay at infinity, namely, e^{-r}/\sqrt{r} . This property is not robust, and even for very small propagation speeds, the decay at infinity changes to algebraic, one proportional to $1/r^2$. The most disappointing property is, however, the fact that the steadily propagating 2D shapes are not robust (structurally stable) and after some time they either dissipate or blow up. This important result was first found by means of a special new scheme in [48] and confirmed quantitatively with a different numerical approximation [49]. All these findings rule out the strictly localized solutions as QPs. The non-robustness with respect to the parameter representing the phase speed of the structure tells us that in order to make the standing localized wave move, the profile behavior up to infinity has to be *instantaneously* changed, which is hard to imagine as a physically realistic process.

In order to find out if pseudolocalized solutions of Eq. (11) can exist we take $\alpha_3 = 0$ and consider the spherical symmetric stationary solution $u = G(\rho)$:

$$\Delta_\rho[G - \alpha_2 G^2 - \beta^2 \Delta_\rho G] = 0, \quad (12)$$

where Δ_ρ is the Laplace operator in spherical coordinates, already defined in Eq. (8). The last equation can be integrated once to obtain

$$G - \alpha_2 G^2 - \beta^2 \frac{1}{\rho^2} \frac{d}{d\rho} \left[\rho^2 \frac{dG}{d\rho} \right] = \frac{C_1}{\rho} + C_2. \quad (13)$$

In order that the solution decays at infinity we set $C_2 = 0$. Now, setting $C_1 = 0$ gives the strictly localized solution which has been discussed in previous paragraphs as not suitable for our purposes. Respectively, $C_1 \neq 0$ leads to a pseudo-localized solution. It is important to note that the linearized version of the last equation has the same solution given in Eq. (6) for the shell-equation (4). Let us call the last solution “peakon” borrowing the terminology introduced in [50] in a different physical situation.

A special numerical investigation of the role of the non-linearity for defining the shape of the pseudolocalized solution was recently conducted in [51]. The result is that the inclusion of nonlinearity does not qualitatively change the profile given in Eq. (6). The nonlinearity can make the profile slightly steeper at the origin, but the asymptotic behavior that is essential for the interaction of two QPs remains unchanged. All this means that, even in the case of a generic Boussinesq equation, the pseudolocalized solutions exist and are valid candidates for QPs. Actually, in this case, the solution $\Phi(\rho)$ is even a better quantitative approximation for the profile of the QP, hence it can be used as the basis for a coarse grain description of the field.

The amplitude $\Phi(\rho)$ can be interpreted as the elevation of the middle 3D surface of a shell (say, along a fourth dimension). To an observer confined to the middle 3D surface it would appear as some kind of density. For this

reason, we use a density plot in Fig. 2. In order to see the internal structure more clearly, we present only half of the ball corresponding to the main part of the peakon. Naturally, the latter is not exactly a ball, because it extends to infinity but the density is smaller far from the center, so if we limit the number of contours to a certain value (sphere), the outermost contour shown is a ball.

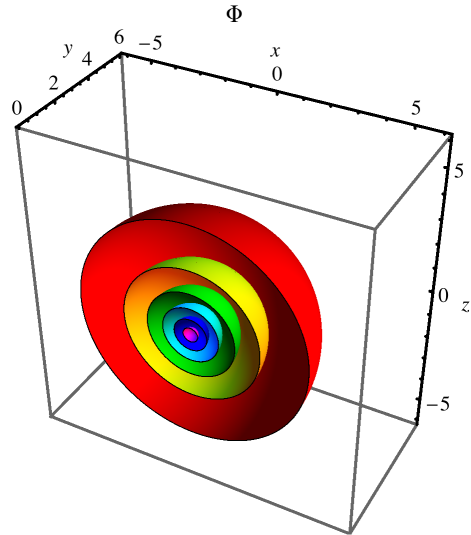


FIG. 2: Resected ball representing a single radial peakon for $\beta = 1$.

IV. COORDINATE SYSTEM FOR STEADILY PROPAGATING LOCALIZED WAVES

The full three-dimensional description of the interaction of QPs of the pseudolocalized type requires significant computational resources. It is well known in 1D that when propagating, the QPs experience contraction or, in some cases, elongation of the overall scale of the support. In 2D, the change of shape is more intricate (see [52]), leading to relative contraction in the direction of motion, on the background of overall elongation of the scales of the solution. The first decisive simplification here will be to assume that when propagating, the QPs do not actually change their shapes, retaining the spherical symmetry. This is a typical assumption when applying the VA.

The second important simplification is to consider QPs which interact only along the line connecting their centers. Since we have already assumed that the shape retains the spherical symmetry during the interaction, this is not really an additional restriction on the solution. Even if the QPs move with arbitrary trajectories in the 3D configurational space, the force acting between them will depend only on the distance between the QPs, as measured along the line connecting their centers. For this reason we restrict ourselves to the case when the QPs move only in one of the directions, say the z -axis. Then

we can consider the radially symmetric cylindrical coordinates $r = \sqrt{x^2 + y^2}$ and z . Their relation to the spherical coordinates is $\rho := \sqrt{r^2 + z^2}$ and $\theta = \arctan(z/r)$. Then

$$r = \rho \cos \theta, \quad z = \rho \sin \theta.$$

The cylindrically symmetric Laplace operator is given by

$$\Delta := \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}. \quad (14)$$

V. COARSE-GRAIN DESCRIPTION USING THE FIELD OF A SINGLE QP

For a localized wave moving in the z -direction according to the law of motion $z = Z(t)$, without changing its shape, one has

$$u(r, z, t) = A\Phi(\sqrt{r^2 + [z - Z(t)]^2}), \quad (15)$$

where A is its amplitude. At this stage we do not consider the changes of the effective support of the wave (pseudo-Lorentzian contraction or elongation) associated with the motion. In other words, the speed $\dot{Z}(t)$ is small in comparison with unity. Then

$$u_t = A\Phi'(\sqrt{r^2 + [z - Z(t)]^2}) \frac{\dot{Z}(z - Z)}{\sqrt{r^2 + (z - Z)^2}}. \quad (16)$$

Consequently, the integral of the kinetic energy over the space is:

$$\begin{aligned} & \frac{1}{2} \iiint_{-\infty}^{\infty} [(u_t)^2] r dr dz \\ &= \pi (\dot{Z})^2 A^2 \iiint_{-\infty}^{\infty} (\Phi')^2 \frac{(z - Z)^2}{r^2 + (z - Z)^2} r dr dz \\ &= \pi (\dot{Z})^2 A^2 \int_0^{\infty} \int_0^{\pi} [\Phi'(\rho)]^2 \rho^2 \cos^2 \theta \sin \theta d\rho d\theta \\ &= \frac{2\pi}{3} (\dot{Z})^2 A^2 \int_0^{\infty} [\Phi'(\rho)]^2 \rho^2 d\rho = \frac{1}{2} \mathbf{m} A^2 \dot{Z}^2, \\ & \text{where } \mathbf{m} := \frac{4\pi}{3} \int_0^{\infty} [\Phi'(\rho)]^2 \rho^2 d\rho = \frac{2\pi}{3\beta}. \end{aligned} \quad (17)$$

Note that here the relation $z - Z = \rho \sin \theta$ is used in manipulating the integrals. In the last line of Eq. (17), Eq. (7) is also acknowledged. Respectively,

$$\nabla u = A\Phi'(\sqrt{r^2 + [z - Z(t)]^2}) \frac{(r, z - Z)}{\sqrt{r^2 + (z - Z)^2}}, \quad (18)$$

where $(r, z - Z)$ is the vector with the respective components. Then it follows from Eq. (7)

$$\iiint_{-\infty}^{\infty} \frac{1}{2} (\nabla u)^2 dx dy dz = \frac{\pi A^2}{\beta}. \quad (19)$$

In similar fashion from Eq. (8) we obtain

$$\frac{\beta^2}{2} \iiint_{-\infty}^{\infty} (\Delta u)^2 dx dy dz = \frac{\pi A^2}{\beta}. \quad (20)$$

Now for the discrete Lagrangian we obtain

$$\mathbb{L} = \frac{1}{2} \mathbf{m} A^2 \dot{Z}^2 - \frac{\pi A^2}{\beta} - \frac{\pi A^2}{\beta}, \quad (21)$$

and the Euler-Lagrange equation for the collective variable $Z(t)$ reads

$$\frac{d}{dt} [\mathbf{m} \frac{d}{dt} Z] = 0, \quad (22)$$

which is nothing else but Newton's law of inertia. This suggests that we interpret the quantity \mathbf{m} as the mass of the QP concentrated in the center of the localized wave. Thus, we call \mathbf{m} the *pseudomass* or the “wave mass” of a QP. The “effective” mass $\mathbf{m} A^2$ is then simply proportional to the square of the amplitude of the quasi-particle (see Eq. (21)).

Since, the solutions can have both positive and negative amplitude, one can define also a quantity called *pseudovolume* as the integral of the curvature

$$\mathbf{v} := -2\pi \int_0^{\infty} \int_0^{\pi} A \Delta \Phi(\rho) \rho^2 \sin \theta d\rho d\theta = 4A\pi \quad (23)$$

This means that due to the convention with the minus sign, in the case of positive amplitude $A > 0$, we have a positive *pseudovolume*. It is interesting to note that the magnitude of β does not affect the numerical value of the *pseudovolume*. The latter is just a number which does not depend on β .

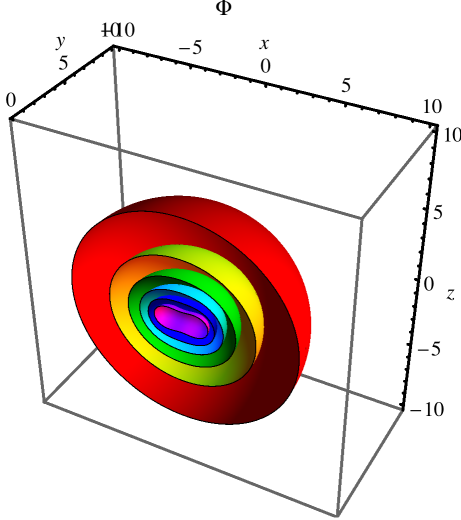
The introduction of the pseudovolume allows us to understand better the notion of anti-particle. The mass of an anti-quasi-particle (anti-QP) is positive, while the volume is negative. In the response to a force, the anti-QP (negative volume) will behave as matter but when it overlaps with a QP (positive volume) they will cancel over each other.

VI. COARSE-GRAIN DESCRIPTION OF THE FIELD OF TWO PSEUDOLocalized QPS

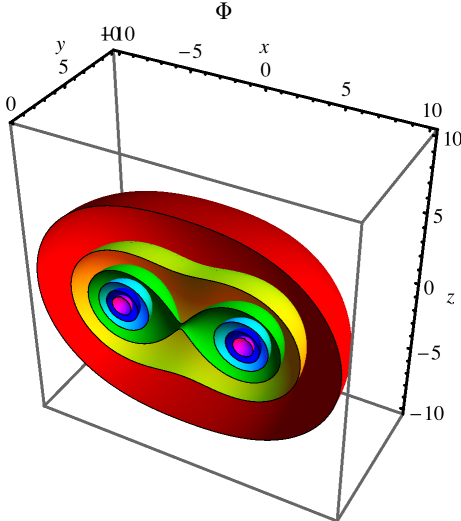
The main idea of VA is to use a reasonable approximation for the shape of the one-wave solution and then, from the continuous Lagrangian, derive the discrete Lagrangian and the laws of motion for the centers of the QPs involved. When a CGD is attempted, the shapes of the QPs are taken to be the shape of a single, stationary, propagating solitary wave. For the case under consideration, we stipulate that

$$u = A_L \Phi(\rho_L) + A_R \Phi(\rho_R), \quad \rho_k := \sqrt{r^2 + (z - Z_k)^2}, \quad (24)$$

where k assumes the values L for the left QP and R for the right QP. Here the constraint is acknowledged that the QPs are supposed to move only along the axes that connect their centers (z -axis in the adopted notations). The case of two equal radial peakons ($A_L = A_R = 1$) with $\beta = 1$ is depicted in Fig. 3 for two different distances between their centers.



(a) distance between centers is 1.18.



(b) distance between centers is 4.

FIG. 3: Two radial peakons for $A_L = A_R = 1$ and two different distances between their centers. $\beta = 1$

In order not to obscure the main idea, we will not

consider here the dependence of shape on the phase speed $c_i := \dot{Z}_i(t)$. This amounts to the assumption that the phase speeds are small. Such an assumption is strictly valid only at the initial stage after the QPs are allowed to move under the forces that arise due to the presence of other QPs.

For the time derivative of the function u from Eq. (24), we find (see the one-QP case from Eq. (16)):

$$u_t = -A_L \Phi'(\rho_L) \frac{\dot{Z}_L(z - Z_L)}{\rho_L} - A_R \Phi'(\rho_R) \frac{\dot{Z}_R(z - Z_R)}{\rho_R}. \quad (25)$$

In some places we suppress the explicit time argument of $Z_{L,R}$ for the sake of clarity. Similarly, for the spatial derivatives we have

$$u_z = A_L \Phi'(\rho_L) \frac{(z - Z_L)}{\rho_L} + A_R \Phi'(\rho_R) \frac{(z - Z_R)}{\rho_R}, \quad (26)$$

$$u_r = A_L \Phi'(\rho_L) \frac{r}{\rho_L} + A_R \Phi'(\rho_R) \frac{r}{\rho_R}. \quad (27)$$

A. Potential due to the Laplacian term

Now, turning to the potential of interaction we acknowledge that there are two terms in the Lagrangian, containing ∇u and Δu , respectively. For the former we have

$$\begin{aligned} \pi \iint [(u_r)^2 + (u_z)^2] r dr dz &= \pi \frac{A_L^2 + A_R^2}{\beta} + A_L A_R V_a(w), \\ V_a(w) &:= 2\pi \iint_{-\infty}^{\infty} \frac{r^2 + (z - Z_L)(z - Z_R)}{\rho_L \rho_R} \\ &\quad \times \Phi'(\rho_L) \Phi'(\rho_R) r dr dz. \end{aligned} \quad (28)$$

Let us now introduce a new variable $\zeta = z - Z_L$ and the notation $w = Z_R - Z_L$. Respectively, we keep the notation ρ for $\rho_L = \sqrt{r^2 + \zeta^2}$ and $\rho_R = \sqrt{r^2 + (\zeta - w)^2} = \sqrt{\rho^2 - 2\rho w \cos \theta + w^2}$. Then the potential due to the Laplacian term is

$$V_a(w) = 2\pi \int_0^\infty \int_0^\pi \frac{\rho^2 - \rho w \cos \theta}{\rho \rho_R} \Phi'(\rho) \Phi'(\rho_R) \rho^2 \sin \theta d\rho d\theta. \quad (29)$$

After performing the integration and applying some standard manipulations, we obtain

$$\begin{aligned} V_a(w) &= \frac{\pi}{2\beta^2 w} \left[-\gamma (-2 + 2\beta^2 + 2\beta w) e^{-w/\beta} - 2(-1 + \beta^2 - 2\beta w) e^{z/\beta} \text{Ei}\left(-\frac{2w}{\beta}\right) + (-2 + 2\beta^2 - 4\beta w) e^{w/\beta} \text{Ei}\left(-\frac{w}{\beta}\right) \right. \\ &\quad \left. + (-2 + 2\beta^2 + 4\beta w) e^{-w/\beta} \text{Ei}\left(\frac{w}{\beta}\right) + (-2 + 2\beta^2 + 4\beta w) \ln(2\beta) e^{-w/\beta} + (2 - 2\beta^2 - 4\beta w) \ln(w) e^{-w/\beta} \right], \end{aligned} \quad (30)$$

where $\gamma \approx 0.5772156649$ is the Euler constant and $\text{Ei}(w) := \int_{-\infty}^w e^t/t dt$ is the exponential integral.

The result most closely to the interaction integrals which can be found in the literature is given in [53] and elaborated in [54], which is one of the few treatises on the interaction of 2D and 3D solitons. The expressions from the cited works differ from ours because they are for strictly localized QPs. Fig. 4 shows the result. The upper panel presents linear axes, while the lower panel gives the same information in log-log format.

The most important feature of $V_a(w)$ is that it changes sign. This means that it is not uniformly attractive or repelling. It is attractive for moderate to large distances between the QPs and becomes repelling when the particles draw closer. The magnitude of the potential in the repelling mode is several orders of magnitude larger, which means that the repulsion is very strong when the distance between the centers of the QPs falls below the threshold shown in Fig. 5. The negative part of the potential, being significantly stronger than the positive part acts to break a possible ensemble of several QPs (an “nucleos”) and, in this instance resembles the definition of the weak force in physics. Naturally, the much weaker portion of the force acting at very large distances with potential $1/w$ can be called “gravitational” because of its one-to-one correspondence with the respective attraction law in physics.

B. Potential due to biharmonic term

In this subsection we deal with the last term in Eq. (5). To this end we observe that

$$\Omega(\rho) := \Delta\Phi(\rho) = \frac{1}{\rho^2} \frac{\partial}{\partial \rho} \rho^2 \frac{\partial \Phi}{\partial \rho} = -\frac{e^{-\rho/\beta}}{\beta^2 \rho}. \quad (31)$$

Then for the assumed profile consisting of the superposition of two QP’s of the type defined in the Eq. (6) we have

$$\begin{aligned} \frac{\beta^2}{2} \iiint_{-\infty}^{\infty} (\Delta u)^2 dx dy dz &= \pi \frac{A_L^2 + A_R^2}{\beta} + A_L A_R V_b(w), \\ V_b(w) &:= 2\beta^2 \pi \int_{-\infty}^{\infty} \int_0^{\infty} \Omega(\rho) \Omega(\rho_R) r dr dz = \frac{\pi}{\beta} e^{-w/\beta}. \end{aligned} \quad (32)$$

The last expression is obtained by using the expression Eq. (A9). Clearly, the potential due to the bi-harmonic term is positive but not strong enough to change the main behavior $V_a(w)$.

C. Total potential and force

In order to complete the description of the dynamical properties of the system of two QPs, we need the cross-mass (see the next subsection) and total potential

$$V(w) := V_a(w) + V_b(w), \quad (33)$$

as they were defined in and Eq. (29) and Eq. (32). In order to save space, we will not write explicitly the expression for the sum, because both components are well specified above. The total potential is shown in Fig. 4.

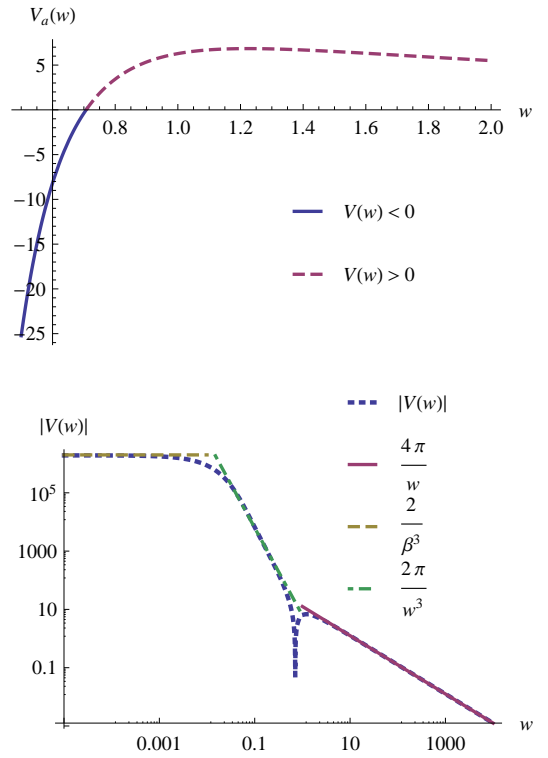


FIG. 4: The full interaction potential for $\beta = 0.01$.

For the total force, Fig. 5 shows its behavior and a comparison to different best fit functions. The result is presented in log-log scale, because of the great disparity of the magnitudes.

$$F(w) := -\frac{dV(w)}{dw} = -\frac{2e^{-w/\beta}\pi}{\beta^2} + \frac{1}{2\beta^2 w^2} \left[-e^{-\frac{w}{\beta}} \gamma(-2 + 2\beta^2 + 2\beta w) - 2e^{\frac{w}{\beta}} (-1 + \beta^2 - 2\beta w) \text{Ei}\left(-\frac{2w}{\beta}\right) - 2e^{-\frac{w}{\beta}} \ln(2\beta) \right]$$

$$\begin{aligned}
& + e^{\frac{w}{\beta}}(-2 + 2\beta^2 - 4\beta w)\text{Ei}\left(-\frac{w}{\beta}\right) + e^{-\frac{w}{\beta}}(-2 + 2\beta^2 + 4\beta w)\text{Ei}\left(\frac{w}{\beta}\right) + e^{-\frac{w}{\beta}}(2 - 2\beta^2 - 4\beta w)\ln(w) \\
& + 2\beta e^{-\frac{w}{\beta}}(\beta + 2w)\ln(2\beta) \Big] - \frac{1}{4\beta^2 w} \Big[-2\beta e^{-\frac{w}{\beta}}\gamma + \frac{(2 - 2\beta^2 - 4\beta w)}{w}e^{-\frac{w}{\beta}} - 4\beta e^{\frac{w}{\beta}}\text{Ei}\left(-\frac{w}{\beta}\right) + \frac{-2 + 2\beta^2 - 4\beta w}{w} \\
& - \frac{2(-1 + \beta^2 - 2\beta w)}{w}e^{-\frac{w}{\beta}} + \frac{\gamma(-2 + 2\beta^2 + 2\beta w)}{\beta}e^{-\frac{w}{\beta}} + \frac{-2 + 2\beta^2 + 4\beta w}{w} + 4\beta e^{\frac{w}{\beta}}\text{Ei}\left(-\frac{2w}{\beta}\right) + 4\beta e^{-\frac{w}{\beta}}\text{Ei}\left(\frac{w}{\beta}\right) \\
& - 2\frac{(-1 + \beta^2 - 2\beta w)}{\beta}e^{\frac{w}{\beta}}\text{Ei}\left(-\frac{2w}{\beta}\right) + \frac{(-2 + 2\beta^2 - 4\beta w)}{\beta}e^{\frac{w}{\beta}}\text{Ei}\left(-\frac{w}{\beta}\right) - \frac{(-2 + 2\beta^2 + 4\beta w)}{\beta}e^{-\frac{w}{\beta}}\text{Ei}\left(\frac{w}{\beta}\right) \\
& + \frac{2}{\beta}e^{-\frac{w}{\beta}}\ln(2\beta) + 4\beta e^{-\frac{w}{\beta}}\ln(2\beta) - 2e^{-\frac{w}{\beta}}(\beta + 2w)\ln(2\beta) - 4\beta e^{-\frac{w}{\beta}}\ln(w) - \frac{(2 - 2\beta^2 - 4\beta w)}{\beta}e^{-\frac{w}{\beta}}\ln(w) \Big]. \quad (34)
\end{aligned}$$

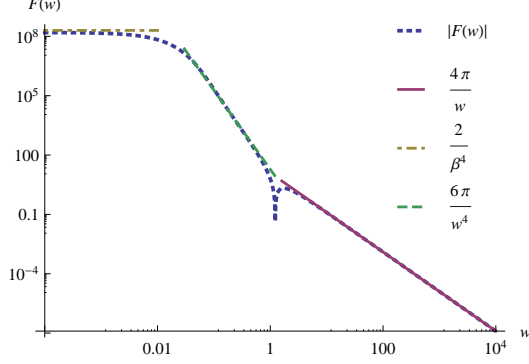


FIG. 5: The force of interaction between two QPs.

D. Crossmass

For the kinetic part of the energy of the system of two QPs we derive

$$\begin{aligned}
\frac{1}{2} \iiint_{-\infty}^{\infty} (u_t)^2 dx dy dz &= \frac{1}{2} \mathbf{m}_L A_L^2 \dot{Z}_L^2 + \frac{1}{2} \mathbf{m}_R A_R^2 \dot{Z}_R^2 \\
&+ \mathbf{m}_{LR} A_L A_R \dot{Z}_L \dot{Z}_R, \quad (35)
\end{aligned}$$

where the pseudomasses are given by Eq. (17), $\mathbf{m}_L = \mathbf{m}_R = \frac{2\pi}{3\beta}$, and along with the pseudomass of each QP we have also the *crossmass* or the *induced mass*:

$$\begin{aligned}
\mathbf{m}_{LR}(w) &:= \iiint_{-\infty}^{\infty} \frac{(z - Z_L)(z - Z_R)}{\rho_L \rho_R} \Phi'(\rho) \Phi'(\rho_R) dx dy dz = 4\pi \int_{-\infty}^{\infty} \int_0^{\infty} \Phi'(\rho_L) \Phi'(\rho_R) \frac{\zeta(\zeta - w)}{\rho_L \rho_R} r dr d\zeta \\
&= 2\pi \int_0^{\infty} \left\{ \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\rho \cos^2 \theta - w \cos \theta}{\sqrt{\rho^2 - 2\rho w \cos \theta + w^2}} \Phi'(\rho_R) \rho \sin \theta d\theta \right\} \Phi'(\rho) \rho d\rho. \quad (36)
\end{aligned}$$

The result of the integration can be found in Eq. (A11), which gives for the crossmass the following expression:

$$\begin{aligned}
\mathbf{m}_{LR}(w) &= \frac{2\pi}{\beta w^3} [e^{-w/\beta} (4\beta^3 + 4\beta^2 w \\
&+ 2\beta w^2 + w^3) - 4\beta^3]. \quad (37)
\end{aligned}$$

Apparently, the fact the the kinetic energy Eq. (35) contains additional terms in the case of two QPs was first noticed in [55]. The term *crossmass* was coined in [27] for the case of interacting solitons in the case sGE and the pseudo-Newtonian law of interaction of two QPs was elaborated in full detail therein as well, including a numerical solution for the trajectories. Here we follow the line of reasoning from [27].

Fig. 6 depicts the dependence of the crossmass on the distance between the particles for the above chosen dis-

person parameter $\beta = 0.01$.

VII. LAW OF MOTION FOR TWO INTERACTING QPS

Now we move to the essence of the coarse-grain description, namely, replacing in the original Lagrangian of the system Eq. (5) the sum of Eqs. (28), (32), and Eq. (35). As a result after acknowledging Eq. (33) we obtain the following coarse-grain (discrete) Lagrangian

$$\begin{aligned}
\mathbb{L} &= -2\pi \frac{A_L^2 + A_R^2}{\beta} - A_L A_R V(|Z_R - Z_L|) \\
&+ \frac{1}{2} \mathbf{m}_L A_L^2 \dot{Z}_L^2 + \frac{1}{2} \mathbf{m}_R A_R^2 \dot{Z}_R^2 + \mathbf{m}_{LR} A_L A_R \dot{Z}_L \dot{Z}_R \quad (38)
\end{aligned}$$

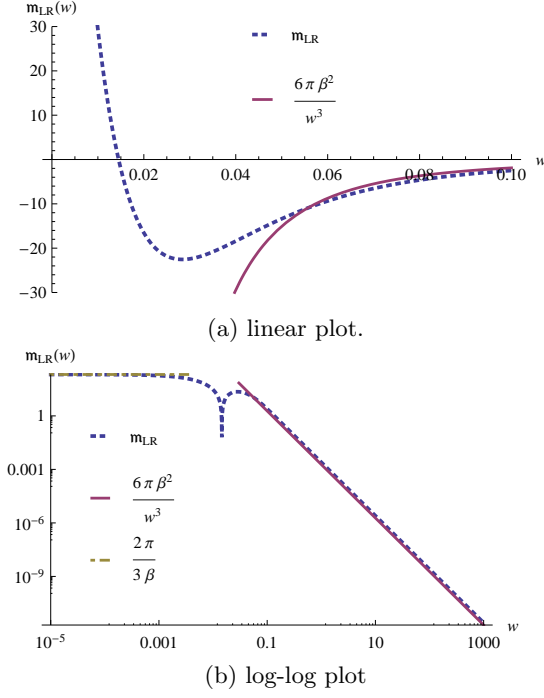


FIG. 6: Dependence of the crossmass on the distance w between the two QPs.

Note that all the terms on the first line of the above equality are constants, so only the terms on the second line contribute to the Euler-Lagrange derivatives with respect to the functions Z_L and Z_R , namely

$$\frac{\delta \mathbb{L}}{\delta Z_L} = A_L^2 \frac{d}{dt} \left[m_L \frac{dZ_L}{dt} \right] + A_L A_R \frac{d}{dt} \left[m_{LR} \frac{dZ_R}{dt} \right] + A_L A_R F(|Z_R - Z_L|), \quad (39a)$$

$$\frac{\delta \mathbb{L}}{\delta Z_R} = A_L A_R \frac{d}{dt} \left[m_{LR} \frac{dZ_L}{dt} \right] + A_R^2 \frac{d}{dt} \left[m_R \frac{dZ_R}{dt} \right] - A_L A_R F(|Z_R - Z_L|), \quad (39b)$$

where F is defined in Eq. (34).

The dynamical system in Eqs. (39) for the motion of the QPs (the centers Z_L and Z_R) differs from one governed by Newton's law of inertia due to the presence of the crossmass terms. These terms indicate that the inertia of a QP is modified by the presence in its vicinity of another accelerating particle. As Mach put it [56]: every interaction in the Universe depends on all the surrounding matter. For reason that should be now obvious to the reader we will refer to Eqs. (39) as *pseudo-Newtonian* law.

As seen in Fig. 6, the crossmass decays with the inverse of the cube of the distance between the QPs, which means that it becomes negligible even for QPs that are not very far from each other and the law of acceleration is practically a Newtonian one. At closer distances, however, the crossmass terms can have a significantly impact on the law of motion.

We solve the equations of motion (39) numerically, in order to understand the mechanism of interaction between the QPs if left to the sole action of the force acting between two of them.

VIII. RESULTS AND DISCUSSION

For definiteness we chose $\beta = 0.01$. This value is not very small, and there should be no problems with very thin layers in which the force varies drastically with slopes $1/w$ or $1/w^4$, respectively. On the other hand, this $\beta = 0.01$ is small enough to give a realistic picture of possible application to wave mechanics where β is extremely small (of order of the square root of Planck's constant).

We chose for the initial positions of the left and right QP to be at -10 and 10 , respectively. For such a separation, the perturbation due to the other particle at the site of the first particle will be small enough, hence the initial speeds can be set equal to zero: $\dot{Z}_L(0) = \dot{Z}_R(0)$. The first experiment is for amplitudes $A_L = A_R = 1$. Fig. 7 shows the result from the numerical integration of Eqs. (39) for the selected set of parameters. If we find

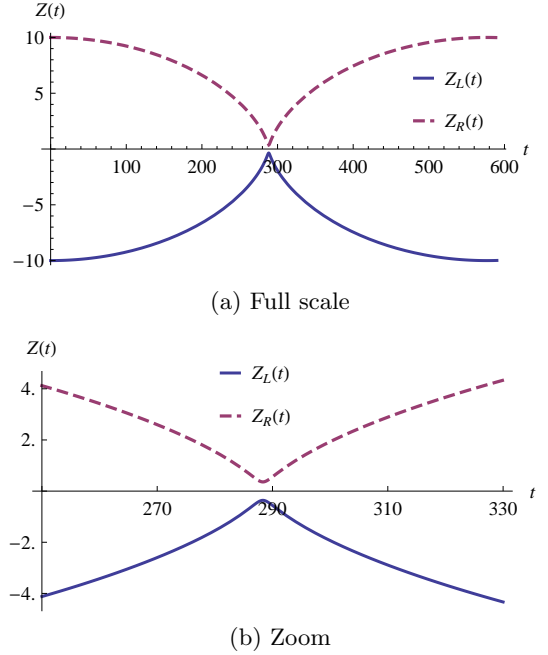


FIG. 7: Trajectories of QPs for $A_L = A_R = 1, \beta = 0.01$ and $Z_R(0) = -Z_L(0) = 10, \dot{Z}_L(0) = \dot{Z}_R(0) = 0$. Solid line: $Z_L(t)$; dashed line: $Z_R(t)$.

the maximum of the modulus of each QP they are equal to each other within the precision of the computations (see Fig. 8). So, QPs of the same masses (same amplitudes) scatter with the same velocities after the collision. The speeds of the QPs increase up to time approximately $t = 715$ after which precipitously decrease under the action of the negative force. After that then begin increasing again in modulus but with changed signs. This is

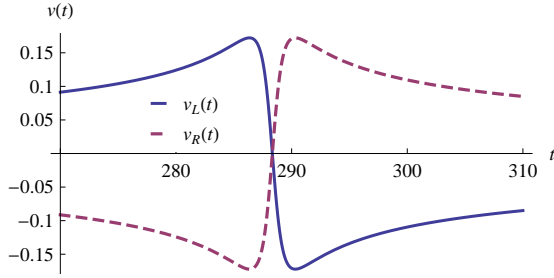


FIG. 8: Zoom of the velocities QPs for the case from Fig. 7. Solid line: $v_L(t)$; dashed line: $v_R(t)$.

a classical behavior for reflecting particles and is stipulated in every model of elementary particles. Here, it stems from the model of QP.

The next experiment is to consider QPs with different amplitudes. We select $A_L = 1$ and $A_R = 2$ (with all other parameters equal) and show the result in Fig. 9. Note that since the pseudomasses depend on the squares of the respective amplitudes, the mass of the left-going particle (initially on the right for positive abscissa) is four times bigger than the mass of the right-going particle (initially on the left). Now it is clear that the more

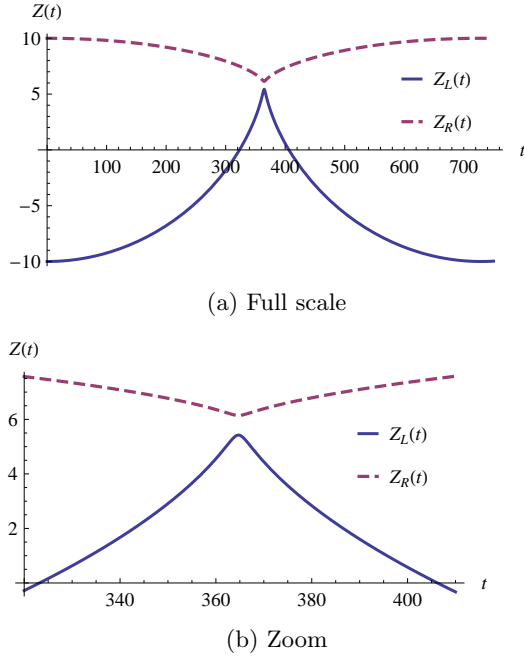


FIG. 9: Trajectories of QPs for $A_L = 2$, $A_R = 1$, $\beta = 0.01$ and $Z_R(0) = -Z_L(0) = 10$, $\dot{Z}_L(0) = \dot{Z}_R(0) = 0$. Solid line: $Z_L(t)$; dashed line: $Z_R(t)$.

massive particle (represented by Z_R) is much less affected by the force and is moving much slower than the particle Z_L (the lighter one). For this reason the two QPs do not collide in the center of the coordinate system but at distance approximately 6. This means that the left particle travel 16 spatial units when the right (heavier) particle

travel just 4 spatial units. This give a rough estimate of the velocities as 4 times faster for the lighter particle whose mass is four times smaller. This is in exact inverse proportion to the ratio of the masses.

In order to assess quantitatively the magnitudes of velocities of the incoming and outgoing QPs we found the maxima of the absolute values of the velocities. The graph of the velocities of the QPs is presented in this case in Fig. 8 Our numerical solution shows that with

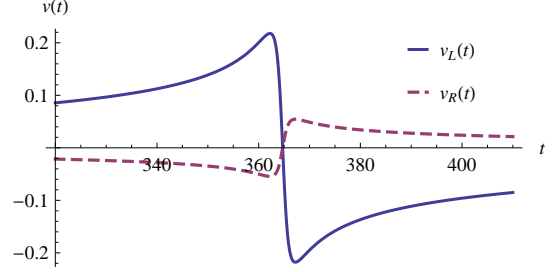


FIG. 10: Zoom of the velocities QPs for the case from Fig. 9. Solid line: $v_L(t)$; dashed line: $v_R(t)$.

high accuracy, both QPs attain maximal moduli of their velocities at approximately $t = 363$ and for this moment of time $v_L(363) = 0.209248$ while $v_R(363) = -0.0523124$. These maximal speeds (moduli of the velocities) can be assumed as the speeds at the cross section of collision. The ratio of the two speeds of reflection with 12 digit rounding is 3.99997689628 which deviates from the expected value 4 by $5.78 \cdot 10^{-6}$. This can be considered as a perfect match with the law of reflection of particles.

Finally, we would like to address the issue of interacting of a QP and an anti-QP. Let set for definiteness that $A_L = 1$ and $A_R = -1$. From Eq. (34) (see also Fig. 5) it is evident that $A_L A_R F(w) < 0$ for large w , i.e. the QP and the anti-QP repel each other. This means that for an initial condition with zero initial velocity the QPs will gradually move farther from each other. In the region of intermediate w , $A_L A_R F(w) > 0$, which means that the force is attractive. Yet, to reach the stage when the QPs begin to attract each other, sufficiently large velocities are to be imposed so the inertia can carry the particle through the border of the region of repelling force. Fig. 5 shows that the change of the sign of the force around $w \approx 1$, and it is positive down to $w \approx 0.02$, while for smaller w it is negative again. Imposing large initial velocity can be called “smashing” a QP into an anti-QP.

The coupling to electromagnetic phenomena has to be added to the model in order to assess its relevance to physical reality. Yet, many of the essential features of the interaction are captured by this conceptually straightforward model based on the coarse-grain description. An enlightening discussion of the concept of anti-particle from the point of view of soliton theory is presented in [57]. The repulsion between matter (QPs) and antimatter (anti-QPs) means that a spontaneous annihilation can be a rare event. A special effort has to be made to smash

a QP and an anti-QP with very high velocity so the barrier of attractive force can be reached. When far enough, the QPs and anti-QPs steer clear from each other.

IX. CONCLUSION

In the present paper a new field equation is explored using the coarse-grain description (CGD), i.e., by identifying individual elements in the field called quasi-particles (QPs).

The solitary wave solutions of the new equation are shown to be pseudolocalized in the sense that the wave profile itself is not in $L^2(\mathbb{R}^3)$ (square integrable functions), but its gradient and Laplacian do belong to $L^2(\mathbb{R}^3)$ space. It is shown that the analytical solution of the linearized equation is a very good quantitative approximation for the shape of the solitary wave, which offers the possibility to identify the interaction analytically.

The pseudolocalized solutions are used as the shapes of the QPs of the new field equation and a CGD is constructed via the so-called variational approximation (VA) in which a discrete Lagrangian is obtained for the trajectory functions of the centers of the QPs. To this end, the interaction potential is computed and shown to decay as the inverse of the distance between the QPs. Respectively, the force depends on the inverse square of the distance between the QPs. This is radically different from the currently known QPs for which the interaction potential (and the force) decay exponentially with the separation of the QPs. Thus, the first QPs that interact “gravitationally” are presented here.

The force of interaction found here is not a monotone function of the distance between the QPs. While at moderate and large distances it is attractive, at smaller distance it becomes repulsive. This is not an intuitive result from the point of view that the shape of each QP itself is monotone. The explanation lies in the fact that the integrals defining the force are over a three-dimensional domain, and while the QPs are closer, the attraction from the surrounding field created by the profile can overwhelm the attraction from the closest elevations. The repulsive part of the force shows a conspicuous resemblance to the so-called weak force. The difference in our inter-

pretation is that this is not another force, but the same force between the QPs which is positive (gravitational) at large distances, and negative (repulsive) at closer distances with much higher magnitude.

From the discrete Lagrangian the laws of motion for two interacting QPs are derived. The notion of wave mass (*pseudomass*) is introduced and an induced mass (*crossmass*) is found alongside the standard mass. The interpretation is that the inertia of a QP is influenced by the presence of an accelerating QP in its vicinity, which fits well in Mach’s notion of all bodies in the Universe being connected to each other.

Similarly to the kink and anti-kink in sGE, the QPs of the new equation come with positive or negative amplitude. We have defined a quantity called the *pseudovolume* which is the integral of the curvature of the profile. It turns out that the pseudo-volume is positive when the amplitude of the respective QP is positive. The force acting between QPs is proportional to the product of their amplitudes, and hence a QP and anti-QP repulse each other if they are at moderate to large distances. At the same time the masses of the two QPs are always positive because they are proportional to the squares of their amplitudes. The anti-QP reacts to non-gravitational forces in the same manner as the QP, but the difference shows only in the binary interaction between an anti-QP and a QP.

The coupled equations for the laws of motion are solved numerically, and the dynamics are interrogated for different initial velocities of the QPs. The scattering (not passing through each other) of QPs is confirmed, and information on the reflection is gathered.

In summary, the present paper lays a claim that a single field model can be found whose QPs interact in a fashion much closer to the known properties of the interactions of real particles.

Acknowledgments

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- [1] T. H. R. Skyrme, Proc. Roy. Soc. A **262**, 237 (1961).
 - [2] J. K. Perring and T. H. R. Skyrme, Nuclear Physics **31**, 550 (1962).
 - [3] E. Fermi, J. R. Pasta, and S. M. Ulam, in *Collected Papers of Enrico Fermi, vol.2* (Univ. Chicago Press, Chicago, 1965), pp. 978–988.
 - [4] N. J. Zabusky and M. D. Kruskal, Phys. Rev. Lett. **15**, 57 (1965).
 - [5] V. G. Makhankov, Phys. Reports **35**, 1 (1978).
 - [6] Y. S. Kivshar and B. A. Malomed, Rev. of Mod. Phys. **61**, 763 (1989).
 - [7] P. Kevrekidis and M. Weinstein, Physica D **142**, 113 (2000).
 - [8] G. A. Maugin, Mech. Res. Commun. **38**, 341 (2011).
 - [9] A. C. Scott, F. Y. F. Chu, and D. W. McLaughlin, Proc. IEEE **61**, 1443 (1973).
 - [10] A. Degasperis, Am. J. Phys. **66**, 486 (1998).
 - [11] M. J. Ablowitz and P. A. Clarkson, *Solitons, Nonlinear Evolution Equations and Inverse Scattering* (Cambridge University Press, Philadelphia, 1992).

- [12] E. Yombaz, Phys. Scr. **73**, 113 (2006).
- [13] R. K. Bullough and F. Caudrey, in *Solitons. Topics in Current Physics*, edited by R. K. Bullough and F. Caudrey (Springer, Berlin, 1980).
- [14] R. K. Dodd, J. C. Eilbeck, J. D. Gibbon, and H. C. Morris, *Solitons and Nonlinear Wave Equations* (Academic Press, New York, 1982).
- [15] T. Yoneyama, Prog. Theor. Phys., Progress Letters **71**, 843 (1984).
- [16] G. Bowtell and A. E. G. Stuart, Phys. Rev. D **15**, 3580 (1977).
- [17] O. Babelon and D. Bernard, Phys. Lett. A **317**, 363 (1993).
- [18] N. S. Manton, Nonlinearity **21**, 7221 (2008).
- [19] T. Matsuda, Lett. Nuovo Cim. **24**, 207 (1979).
- [20] T. Sugiyama, Prog. Theor. Phys. **61**, 1550 (1979).
- [21] D. K. Campbell, J. F. Schonfeld, and C. A. Wingate, Physica D **9**, 1 (1983).
- [22] C. D. Ferguson and C. R. Willis, Physica D **119**, 283 (1998).
- [23] M. J. Rice, Phys. Rev. B **28**, 3587 (1983).
- [24] D. J. Bergman, E. Ben-Jacob, Y. Imry, and E. Maki, Phys. Rev. A **27**, 3345 (1983).
- [25] R. Boesch, P. Stancioff, and C. R. Willis, Phys. Rev. B **38**, 6713 (1988).
- [26] I. Christov and C. I. Christov, Discrete and Continuous Dynamical Systems (DCDS) Supplement, 171 (2009), URL <http://www.aims sciences.org/journals/displayArticles.jsp?paperID=4599>.
- [27] I. Christov and C. I. Christov, Phys. Lett. A **372**, 841 (2008).
- [28] H. D. Nguyen, J. Nonln. Math. Phys. **11**, 180 (2004).
- [29] N. Benes, A. Kasman, and K. Young, J. Nonln. Sci. **16**, 179 (2006).
- [30] G. P. Agrawal, *Fiber-optic communication systems* (Wiley, New York, 1997), 2nd ed.
- [31] Y. Silberberg, Opt. Lett. **15** (1990).
- [32] D. E. Edmundson and R. H. Enns, Phys. Rev. A. **51**, 2491 (1995).
- [33] D. Anderson, Phys. Rev. A **27**, 3135 (1983).
- [34] V. I. Karpman and V. V. Solov'ev, Physica **3D**, 487 (1981).
- [35] P. D. Miller and P. L. Christiansen, Phys. Scripta **61**, 518 (2000).
- [36] U. A. Khawaja, Phys. Rev. E **81**, 056603 (2010).
- [37] P. Rosenau and E. Kashdan, Phys. Rev. Lett. **101**, 264101 (2008).
- [38] P. Rosenau and E. Kashdan, Phys. Rev. Lett. **104**, 034101 (2010).
- [39] E. Schrödinger, Annalen der Physik **79**, 361 (1926).
- [40] C. I. Christov, Math. Comp. Simul. **80**, 91 (2009).
- [41] L. A. Segel, *Mathematics Applied to Continuum Mechanics* (Dover, 1987).
- [42] T. von Kármán, *Festigkeitsprobleme in Maschinenbau* (Teubner, Stuttgart, 1910), vol. IV/4, pp. 311–385.
- [43] P. G. Ciarlet, Arch. Rational Mech. Anal. **73**, 349 (1980).
- [44] J. V. Boussinesq, Journal de Mathématiques Pures et Appliquées **17**, 55 (1872).
- [45] D. J. Benney and J. C. Luke, J. Math. and Phys. **43**, 299 (1964).
- [46] C. I. Christov, Wave Motion **34**, 161 (2001).
- [47] B. B. Kadomtsev and V. I. Petviashvili, Sov. Phys. Dokl. **15**, 539 (1970), translation from Dokl Akad Nauk SSSR 192: 753-756.
- [48] A. Chertock, C. I. Christov, and A. Kurganov, in *Comput. Sci., & High Performance Computing IV, NNFM*, edited by E. Krause (Springer, 2011), pp. 267–281.
- [49] C. I. Christov, N. Kolkovska, and D. Vasileva, in *NMA 2010*, edited by I. Dimov, S. Dimova, and N. Kolkovska (2011), vol. 6046 of *LNCS*, pp. 386–394.
- [50] R. Camassa and D. D. Holm, Phys. Rev. Lett. **71**, 1661 (1993).
- [51] C. I. Christov, Mat. Comp. Simul. (2012), accepted, arXiv:1202.4524.
- [52] C. I. Christov and J. Choudhury, Mech. Res. Commun. **38**, 274 (2011).
- [53] B. Malomed, Phys. Rev. E **58**, 7928 (1998).
- [54] A. Maimistov, B. Malomed, and A. Desyatnikov, Phys. Lett. A **254**, 179 (1999).
- [55] C. I. Christov, in *Continuum Models and Discrete Systems, vol. 1*, edited by G. A. Maugin (Longman, London, 1990), pp. 232–252.
- [56] D. Kleppner and R. J. Kolenkow, *An Introduction to mechanics* (McGraw-Hill, New York, 1973).
- [57] A. Filippov, *The Versatile Soliton* (Springer, New York, 2010).

Appendix A: Evaluation of Integrals

Recall that the shape of a single QP in the linear limit is given by Eq. (6).

1. Potential from the Laplacian term

To evaluate the integral in Eq. (36) Now we can introduce again the spherical coordinates centered at $z = Z_L$.

Then $\rho_L = \rho$ and $\rho_R = \sqrt{\rho^2 - 2\rho w \cos \theta + w^2}$. Then the integral for the potential V can be manipulated to the following

$$\begin{aligned} \int_0^\infty \int_0^\pi \Phi'(\rho) \Phi'(\rho_R) \frac{\rho^2 - \rho w \cos \theta}{\rho \rho_R} \rho^2 \sin \theta d\rho d\theta &= \int_0^\infty \int_0^\pi \Phi'(\rho) \Phi'(\rho_R) \left[\frac{\rho^2 - w^2}{2\rho \rho_R} + \frac{\rho^2 - 2\rho w \cos \theta + w^2}{2\rho \rho_R} \right] \rho^2 \sin \theta d\rho d\theta \\ &= \int_0^\infty \int_0^\pi \frac{1}{2} \Phi'(\rho) \Phi'(\rho_R) \frac{\rho^2 - w^2}{2\rho_R} \rho \sin \theta d\rho d\theta + \int_0^\infty \int_0^\pi \frac{1}{2} \Phi'(\rho) \Phi'(\rho_R) \rho_R \rho \sin \theta d\rho d\theta \quad (\text{A1}) \end{aligned}$$

We observe that

$$\frac{\partial \Phi(\rho_R)}{\partial \theta} = \rho w \Phi'(\rho_R) \frac{\sin \theta}{\rho_R}, \quad \frac{\partial \rho_R}{\partial \theta} = \rho w \frac{\sin \theta}{\rho_R}, \quad (\text{A2})$$

which gives

$$V_a(w) = \frac{1}{2w} \int_0^\infty Q_1(\rho) \Phi'(\rho) (\rho^2 - w^2) d\rho + \int_0^\infty Q_2(\rho) \Phi'(\rho) d\rho, \quad (\text{A3})$$

where the terms that depend on θ are combined in the following integrals (note that $\mu := \cos \theta$):

$$\begin{aligned} Q_1(\rho) &= \int_0^\pi \Phi'(\rho_R) w \frac{\sin \theta}{\rho_R} d\theta = [\Phi(\rho_R)]_{\theta=\pi} - \Phi(\rho_R)|_{\theta=0} \\ &= [\Phi(|\rho + w|) - \Phi(|\rho - w|)] = \left(\frac{1 - e^{-|\rho + w|/\beta}}{|\rho + w|} - \frac{1 - e^{-|\rho - w|/\beta}}{|\rho - w|} \right), \quad (\text{A4}) \end{aligned}$$

$$\begin{aligned} Q_2(\rho) &= \frac{1}{2} \int_0^\pi \Phi'(\rho_R) \rho_R^2 \rho \frac{\sin \theta}{\rho_R} d\theta = \frac{1}{2} \int_0^\pi \left[-\frac{1 - e^{-\rho_R/\beta}}{\rho_R} + \frac{1}{\beta} e^{-\rho_R/\beta} \right] \rho \sin \theta d\theta - \frac{1}{2} \int_0^\pi \frac{1 - e^{-\rho_R/\beta}}{\rho_R} \rho \sin \theta d\theta \\ &\quad + \frac{1}{2} \int_0^\pi \frac{1}{\beta} e^{-\rho_R/\beta} \rho \sin \theta d\theta = -\frac{1}{2w} \int_{|\rho - z|}^{|\rho + z|} [1 - e^{-\rho_R/\beta}] d\rho_R - \frac{1}{2w\beta} \int_{-1}^1 e^{-\frac{1}{\beta} \sqrt{\rho^2 - 2\rho w \mu + w^2}} \rho w d\mu. \quad (\text{A5}) \end{aligned}$$

Proceeding with the manipulations of the last two integrals we obtain

$$\begin{aligned} -\frac{1}{2w} \int_{|\rho - z|}^{|\rho + z|} [1 - e^{-\rho_R/\beta}] d\rho_R &= -\frac{1}{2w} [\rho_R + \frac{1}{\beta} e^{-\rho_R/\beta}]_{|\rho - z|}^{|\rho + z|} - \frac{1}{2w} [|\rho + w| - |\rho - z| + \frac{1}{\beta} (e^{-|\rho + w|/\beta} - e^{-|\rho - w|/\beta})], \\ -\frac{1}{2w\beta} \int_{-1}^1 e^{-\frac{1}{\beta} \sqrt{\rho^2 - 2\rho w \mu + w^2}} \rho w d\mu &= \frac{1}{2w} [e^{-|\rho + w|/\beta} (\beta + \rho + z) - e^{-|\rho - w|/\beta} (\beta + |\rho - w|)]. \end{aligned}$$

Thus, we can provide the final version for Eq. (A5)

$$\begin{aligned} Q_2(w) &= -\frac{1}{2w} (|\rho + w| - |\rho - z|) + \frac{1}{2w} [e^{-|\rho + w|/\beta} (\beta + \rho + z) - e^{-|\rho - w|/\beta} (\beta + |\rho - w|)] \\ &\quad + \frac{1}{w\beta} (e^{-|\rho + w|/\beta} - e^{-|\rho - w|/\beta}). \quad (\text{A6}) \end{aligned}$$

Now,

$$\begin{aligned}
V_a(w) &= \frac{1}{2w} \int_0^\infty \left(\frac{1 - e^{-|\rho+w|/\beta}}{|\rho+w|} - \frac{1 - e^{-|\rho-w|/\beta}}{|\rho-w|} \right) \Phi'(\rho) (\rho^2 - w^2) d\rho \\
&+ \int_0^\infty \left\{ -\frac{1}{2w} (|\rho+w| - |\rho-w|) + \frac{1}{2w} [e^{-|\rho+w|/\beta} (\beta + \rho + z) - e^{-|\rho-w|/\beta} (\beta + |\rho-w|)] + \frac{1}{w\beta} (e^{-|\rho+w|/\beta} - e^{-|\rho-w|/\beta}) \right\} \Phi'(\rho) d\rho \\
&= \frac{e^{-w/\beta} \gamma}{2b^2 w} + \frac{e^{w/\beta} \text{Ei}(-\frac{2w}{\beta})}{2\beta^2 w} - \frac{e^{-w/\beta} \text{Ei}(-\frac{w}{\beta})}{2\beta^2 w} - \frac{e^{-w/\beta} \text{Ei}(\frac{w}{\beta})}{2\beta^2 w} + \frac{e^{-w/\beta} \text{Ei}(-\frac{w}{\beta})}{2\beta^2 w} - \frac{e^{w/\beta} \text{Ei}(-\frac{w}{\beta})}{2\beta^2 w} \\
&\quad - \frac{e^{-w/\beta} (\ln(4) - \ln(\beta^2))}{4\beta^2 w} - \frac{e^{-w/\beta} \ln(\beta)}{\beta^2 w} + \frac{e^{-w/\beta} \ln(w)}{2\beta^2 w}
\end{aligned} \tag{A7}$$

2. Potential from the biharmonic term

Applying the same notations $\rho_L = \rho$, $\rho_R = \sqrt{\rho^2 - 2\rho w \cos \theta + w^2}$ we obtain

$$2\pi\beta^2 \int_{-\infty}^\infty \int_0^\infty \Omega(\rho) \Omega(\rho_R) r dr dz = \frac{2\pi}{\beta^2} \int_0^\infty \left[\int_0^\pi \frac{e^{-\rho_R/\beta}}{\rho_R} \sin \theta d\theta \right] \frac{e^{-\rho/\beta}}{\rho} \rho^2 d\rho. \tag{A8}$$

Using Eq. (A2) we recast the last equation to

$$\begin{aligned}
\frac{2\pi}{\beta^2} \int_0^\infty \frac{1}{\rho w} \left[\int_{|\rho+w|}^{|\rho-w|} e^{-\rho_R/\beta} d\rho_R \right] \frac{e^{-\rho/\beta}}{\rho} \rho^2 d\rho &= \frac{2\pi}{\beta^2} \int_0^\infty [e^{-|\rho-w|/\beta} - e^{-|\rho+w|/\beta}] e^{-\rho/\beta} d\rho \\
&= \frac{2\pi}{\beta^2} \left[-\int_0^\infty e^{-(\rho+w)/\beta} e^{-\rho/\beta} d\rho + \int_0^w e^{-(w-\rho)/\beta} e^{-\rho/\beta} d\rho + \int_w^\infty e^{-(\rho-w)/\beta} e^{-\rho/\beta} d\rho \right] = \frac{2\pi w}{\beta^2} e^{-w/\beta}.
\end{aligned} \tag{A9}$$

3. Crossmass

For the crossmass from Eq. (36), we evaluate the following integral

$$\int_0^\infty \Phi'(\rho) Q_m(\rho, w) \rho d\rho, \quad \text{where} \quad Q_m(\rho, w) = \int_0^\pi \Phi'(\rho_R) \frac{\rho \cos^2 \theta - w \cos \theta}{\sqrt{\rho^2 - 2\rho w \cos \theta + w^2}} \rho \sin \theta d\theta. \tag{A10}$$

Making use of the relation Eq. (A2), we manipulate the inner integral, from the last equation as follows

$$\begin{aligned}
Q_m(\rho, w) &= \int_0^\pi \Phi'(\rho_R) \frac{(\rho \cos^2 \theta - w \cos \theta) \rho \sin \theta}{\sqrt{\rho^2 - 2\rho w \cos \theta + w^2}} d\theta = \frac{1}{w} \left\{ \Phi(\rho_R) (\rho \cos^2 \theta - w \cos \theta) \Big|_0^\pi \right. \\
&+ \int_0^\pi \Phi(\rho_R) (2\rho \cos \theta \sin \theta - w \sin \theta) d\theta \Big\} = \frac{1}{w} \left\{ e^{-(\rho+w)/\beta} - \text{sgn}(\rho - w) e^{-|\rho-w|/\beta} \right. \\
&+ \frac{\beta}{w^2 \rho} \left[(2\beta^2 + 2\beta(w + \rho) + w(w + 2\rho)) e^{|\rho+w|/\beta} - (2\beta|w - \rho| + 2\beta^2 + w(w - 2\rho)) e^{-|w-\rho|/\beta} \right] \Big\}.
\end{aligned}$$

Then

$$\int_0^\infty \Phi'(\rho) Q_m(\rho, w) \rho d\rho = \frac{-4\beta^3 + e^{-w/\beta} (4\beta^3 + 4\beta^2 w + 2\beta w^2 + w^3)}{\beta w^3}. \tag{A11}$$